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SIMPLE SOLUTIONS FOR BUCKLING OF ORTHOTROPIC CONICAL SHELLS

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Abstract—Using Donnell-type shell theory a simple and exact procedure is presented for linear buckling analysis of orthotropic conical shells under axial compressive loads and external pressure. The solution is in the form of a power series in terms of a particularly convenient coordinate system. By analyzing the buckling of a series of conical shells, under various boundary conditions and different material coefficients, the validity of the presented procedure is confirmed.

1. INTRODUCTION

Due to their extensive use, particularly in the aeronautical industry, the buckling of conical shells has been studied by many researchers. Much literature exists on the buckling of isotropic conical shells under compressive axial loads in Seide (1956, 1961), Lackman and Renzien (1960), Singer (1961, 1965), Weigarten *et al.* (1965a, b), Baruch *et al.* (1970) and Tani and Yamaki (1970) and under external pressure in Singer (1961, 1966) and Baruch and Singer (1965) as well as combined loading in Weigarten *et al.* (1965). A simple formula was developed for the buckling of isotropic conical shells by Seide (1956) and later verified by Lackman and Renzien (1960). Seide's formula is independent of boundary conditions and is best used for long shells. Using complex series Singer (1966) and Baruch and Singer (1965) proposed a procedure for solving the three equilibrium equations and two out-of-plane boundary conditions are satisfied identically while the out-of-plane equilibrium equation and in-plane boundary conditions are satisfied approximately. Subsequently, Baruch *et al.* (1970) improved Singer's solution by satisfying the in-plane boundary conditions exactly. Nevertheless the overall solution remained complicated. To our knowledge there has not been a simple solution for buckling analysis of isotropic conical shells under axial loads and external pressure. For orthotropic shells there have been fewer studies. By using the displacement strain relations in Seide (1957), Singer (1962, 1963) derived a set of equations for the buckling of orthotropic conical shells. Following the procedure in Singer (1965), Baruch and Singer (1965) obtained solutions for the buckling of orthotropic conical shells. Baruch's procedure may be used to analyze stiffened conical shells by smearing the stiffeners to find equivalent orthotropic shells. Since this procedure is an extension of the analysis for isotropic shells, it suffers from the same shortcomings mentioned earlier.

In the following we develop a simple and exact procedure for buckling analysis of isotropic and orthotropic conical shells under axial compression and external pressure. The procedure consists of the following steps:

- the buckling equations are developed and expressed in terms of displacements;
- using a new technique, exact solutions are constructed in series form for the governing equations;
- convergence properties of the series solution are determined.

By way of verification, several examples are analyzed and the effects of boundary conditions and elastic coefficients on the buckling loads are investigated.

2. DONNELL-TYPE GOVERNING EQUATIONS

Consider a conical shell as shown in Fig. 1, R_1 and R_2 indicate the radii of the cone at its small and large ends, respectively, α denotes the semi-vertex angle of the cone and L is the cone length along its generator. We now introduce the x–ϕ coordinate system; x is measured along the cone's generator starting at the middle length and ϕ is the circumferential coordinate. The displacements of the shell's middle surface are denoted by U and V along x and ϕ directions respectively, and by W along the normal to the surface (inward positive). In terms of these variables the cone's radius at any point along its length may be expressed as

$$R(x) = R_0 + x \sin \alpha. \tag{1}$$

Now let the cone be subjected to an axially compressive load P and an external normal pressure q. Under this loading the membrane stress resultants, at the critical state, may be expressed as

$$N_{x0} = \frac{P + q\pi(2R_0 + x \sin \alpha)x \sin \alpha}{2\pi R(x) \cos \alpha}$$

$$N_{\phi 0} = \frac{qR(x)}{\cos \alpha}. \tag{2}$$

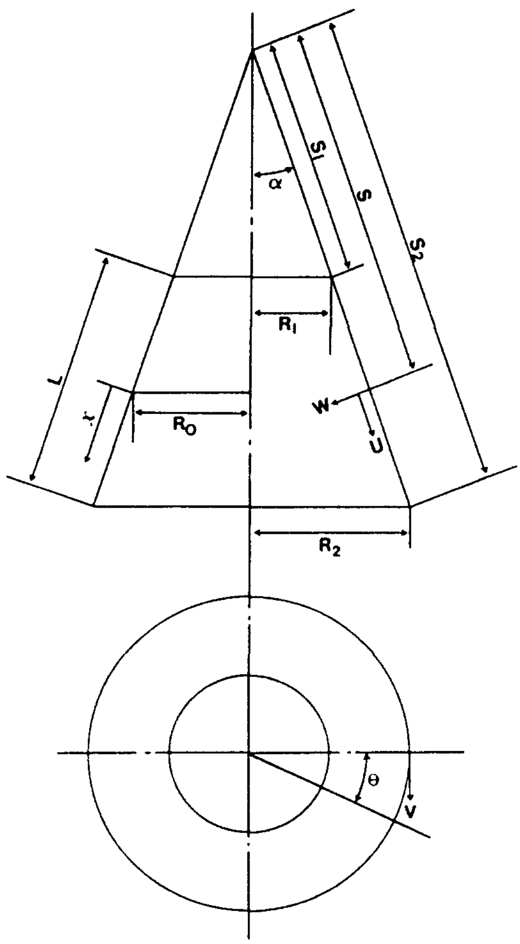

Fig. 1. Geometry and notations for a typical cone.

These equations, based on the membrane theory of shallow shells, degenerate to their more familiar forms for cylindrical shells when α is set equal to zero.

For linear buckling analysis of orthotropic conical shells, under P and q loadings, we adopt the shallow shell theory of Donnell-type and write the governing equations derived in Tong (1988) as

$$\begin{aligned} L_{11}U+L_{12}V+L_{13}W &= 0 \\ L_{21}U+L_{22}V+L_{23}W &= 0 \\ L_{31}U+L_{32}V+L_{33}W+L_NW &= 0 \end{aligned} \tag{3}$$

where

$$\begin{aligned}
L_{11} &= A_{11}\frac{\partial^2}{\partial x^2}+\frac{A_{11}\sin\alpha}{R(x)}\frac{\partial}{\partial x}-\frac{A_{22}\sin^2\alpha}{R^2(x)}+\frac{A_{66}}{R^2(x)}\frac{\partial^2}{\partial\phi^2} \\
L_{12} &= \frac{(A_{12}+A_{66})}{R(x)}\frac{\partial^2}{\partial x\,\partial\phi}-\frac{(A_{22}+A_{66})\sin\alpha}{R^2(x)}\frac{\partial}{\partial\phi} \\
L_{21} &= \frac{(A_{12}+A_{66})}{R(x)}\frac{\partial^2}{\partial x\,\partial\phi}+\frac{(A_{22}+A_{66})\sin\alpha}{R^2(x)}\frac{\partial}{\partial\phi} \\
L_{22} &= A_{66}\left[\frac{\partial^2}{\partial x^2}+\frac{\sin\alpha}{R(x)}\frac{\partial}{\partial x}-\frac{\sin^2\alpha}{R^2(x)}\right]+\frac{A_{22}}{R^2(x)}\frac{\partial^2}{\partial\phi^2} \\
L_{13} &= -\frac{A_{12}\cos\alpha}{R(x)}\frac{\partial}{\partial x}+\frac{A_{22}\sin\alpha\cos\alpha}{R^2(x)} \\
L_{23} &= L_{32} = \frac{A_{22}\cos\alpha}{R^2(x)}\frac{\partial}{\partial\phi} \\
L_{31} &= -\frac{A_{12}\cos\alpha}{R(x)}\frac{\partial}{\partial x}-\frac{A_{22}\sin\alpha\cos\alpha}{R^2(x)} \\
L_{33} &= \frac{A_{22}\cos^2\alpha}{R^2(x)}+D_{11}\frac{\partial^4}{\partial x^4}+\frac{2(D_{12}+2D_{66})}{R^2(x)}\frac{\partial^4}{\partial x^2\,\partial\phi^2}+\frac{D_{22}}{R^4(x)}\frac{\partial^4}{\partial\phi^4} \\
&\quad+\frac{2D_{11}\sin\alpha}{R(x)}\frac{\partial^3}{\partial x^3}-\frac{2(D_{12}+2D_{66})\sin\alpha}{R^3(x)}\frac{\partial^3}{\partial x\,\partial\phi^2}-\frac{D_{22}\sin^2\alpha}{R^2(x)}\frac{\partial^2}{\partial x^2} \\
&\quad+\frac{2(D_{12}+D_{22}+2D_{66})\sin^2\alpha}{R^4(x)}\frac{\partial^2}{\partial\phi^2}+\frac{D_{22}\sin^3\alpha}{R^3(x)}\frac{\partial}{\partial x} \\
L_N &= \frac{1}{R(x)}\frac{\partial}{\partial x}\left[R(x)N_{x0}\frac{\partial}{\partial x}\right]+\frac{1}{R^2(x)}\frac{\partial}{\partial\phi}\left(N_{\phi 0}\frac{\partial}{\partial\phi}\right)
\end{aligned} \tag{4}$$

and A_{ij} and D_{ij} $(i, j = 1, 2, 6)$ are calculated from the following equations:

$$\begin{aligned}
&A_{11} = \frac{E_xh}{1-\mu_{x\phi}\mu_{\phi x}},\quad A_{12} = \frac{\mu_{\phi x}E_xh}{1-\mu_{x\phi}\mu_{\phi x}},\quad A_{21} = \frac{\mu_{x\phi}E_\phi h}{1-\mu_{x\phi}\mu_{\phi x}},\quad A_{22} = \frac{E_\phi h}{1-\mu_{x\phi}\mu_{\phi x}},\quad A_{66} = G_{x\phi}h \\
&D_{11} = \frac{E_xh^3}{12(1-\mu_{x\phi}\mu_{\phi x})},\quad D_{12} = \frac{\mu_{\phi x}E_xh^3}{12(1-\mu_{x\phi}\mu_{\phi x})} \\
&D_{21} = \frac{\mu_{x\phi}E_\phi h^3}{12(1-\mu_{x\phi}\mu_{\phi x})},\quad D_{22} = \frac{E_\phi h^3}{12(1-\mu_{x\phi}\mu_{\phi x})},\quad D_{66} = \frac{G_{x\phi}h^3}{12}
\end{aligned} \tag{5}$$

in which E_x, E_ϕ, $\mu_{x\phi}$, $\mu_{\phi x}$ and $G_{x\phi}$ are material elastic constants; h is the wall thickness and

$$\mu_{x\phi}E_\phi = \mu_{\phi x}E_x.$$

The force and moment stress resultants are expressed in terms of the displacements U, V and W by

$$\begin{Bmatrix} N_x \\ N_\phi \\ N_{x\phi} \\ M_x \\ M_\phi \\ M_{x\phi} \end{Bmatrix} = \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & 0 \\ 0 & 0 & l_{43} \\ 0 & 0 & l_{53} \\ 0 & 0 & l_{63} \end{bmatrix} \begin{Bmatrix} U \\ V \\ W \end{Bmatrix} \tag{6}$$

where

$$l_{i1} = A_{i1}\frac{\partial}{\partial x} + \frac{A_{i2}\sin\alpha}{R(x)}, \quad l_{i2} = \frac{A_{i2}}{R(x)}\frac{\partial}{\partial\phi}, \quad l_{i3} = -\frac{A_{i2}\cos\alpha}{R(x)}$$

$$l_{31} = \frac{A_{66}}{R(x)}\frac{\partial}{\partial\phi}, \quad l_{32} = A_{66}\left(\frac{\partial}{\partial x} - \frac{\sin\alpha}{R(x)}\right)$$

$$l_{j3} = -D_{1i}\frac{\partial^2}{\partial x^2} - \frac{D_{i2}\sin\alpha}{R(x)}\frac{\partial}{\partial x} - \frac{D_{i2}}{R^2(x)}\frac{\partial^2}{\partial\phi^2}, \quad l_{63} = -\frac{D_{66}}{R(x)}\frac{\partial}{\partial x}\left[\frac{1}{R(x)}\frac{\partial}{\partial\phi}\right] \tag{7}$$

where $i = 1, 2$ and $j = 3+i$.

The transverse shear force resultants can be obtained from M_x, M_ϕ and $M_{x\phi}$ by

$$Q_x = \frac{1}{R(x)}\frac{\partial}{\partial x}[R(x)M_x] - \frac{M_\phi\sin\alpha}{R(x)} + \frac{1}{R(x)}\frac{\partial M_{x\phi}}{\partial\phi}$$

$$Q_\phi = \frac{1}{R(x)}\frac{\partial}{\partial x}[R(x)M_{x\phi}] + \frac{M_{x\phi}\sin\alpha}{R(x)} + \frac{1}{R(x)}\frac{\partial M_\phi}{\partial\phi}. \tag{8}$$

The related boundary conditions may be expressed as

$$\begin{aligned} N_x &= 0 \quad \text{or} \quad U = 0 \\ N_{x\phi} &= 0 \quad \text{or} \quad V = 0 \\ M_x &= 0 \quad \text{or} \quad \frac{\partial W}{\partial x} = 0 \quad \text{when} \quad x = \pm\frac{L}{2} \\ Q_x &= 0 \quad \text{or} \quad W = 0. \end{aligned} \tag{9}$$

For simplicity, let us first consider the following two types of boundary conditions in detail:

Case 1: Simply-supported boundary conditions at $x = \pm L/2$.

There exist four subclasses of simply-supported conditions. These are denoted as follows:

$$\begin{aligned} SS_1&: \quad N_{x\phi} = N_x = M_x = W = 0 \\ SS_2&: \quad N_{x\phi} = U = M_x = W = 0 \\ SS_3&: \quad V = N_x = M_x = W = 0 \\ SS_4&: \quad V = U = M_x = W = 0. \end{aligned} \tag{10}$$

Case 2: Clamped boundary conditions at $x = \pm L/2$.
The four subclasses in this case are:

$$\mathrm{CC}_1:\quad N_{x\phi} = N_x = \frac{\partial W}{\partial x} = W = 0$$

$$\mathrm{CC}_2:\quad N_{x\phi} = U = \frac{\partial W}{\partial x} = W = 0$$

$$\mathrm{CC}_3:\quad V = N_x = \frac{\partial W}{\partial x} = W = 0$$

$$\mathrm{CC}_4:\quad V = U = \frac{\partial W}{\partial x} = W = 0. \tag{11}$$

The above set of governing equations degenerate to those of cylindrical shells when α is set equal to zero. It is also worth noting that if the starting point of the x-axis is changed to the cone's vertex, where the radius is equal to zero, the x–ϕ coordinate system will coincide with the s–ϕ coordinate system used by many previous researchers. A further point of interest is the case when α, the semi-vertex angle, approaches a right angle. In this case, the differential operators L_{13}, L_{23}, L_{31} and L_{32} approach zero and the three equilibrium equations become independent, that is, the first two equations will then describe the in-plane problem and the third the buckling problem of circular plates under axially symmetric in-plane loading.

Evidently the system of governing equations presented in the foregoing is complex and to our knowledge exact solutions have not been given for these equations. In the following section we outline a strategy for constructing general solutions for these equations.

3. EXACT SOLUTIONS

An inspection of the differential operators $L_{i,j}$ $(i, j = 1, 2, 3)$ and L_N in eqns (4) reveals the following properties:

The coefficients of all these operators are functions of x only, i.e. they are independent of ϕ, and they include terms of the following form: $1/R^k(x)$, $k = 0, 1, 2, 3, 4$. For the operators $L_{i,j}$ $(i = 1, 2; j = 1, 2, 3)$, k takes values from zero to two. For L_{3j} $(j = 1, 2, 3)$ the value of k ranges from zero to four. These useful properties allow us to change the equations into a more convenient form. Multiplying the first two equations of (3) by $R^2(x)$ and the third by $R^4(x)$ we obtain the following modified equations:

$$\begin{aligned} &L^*_{11}U + L^*_{12}V + L^*_{13}W = 0 \\ &L^*_{21}U + L^*_{22}V + L^*_{23}W = 0 \\ &L^*_{31}U + L^*_{32}V + L^*_{33}W + L^*_N W = 0 \end{aligned} \tag{12}$$

where

$$L^*_{11} = A_{11}R^2(x)\frac{\partial^2}{\partial x^2} + A_{11}R(x)\sin\alpha\frac{\partial}{\partial x} - A_{22}\sin^2\alpha + A_{66}\frac{\partial^2}{\partial \phi^2}$$

$$L^*_{12} = (A_{12}+A_{66})R(x)\frac{\partial^2}{\partial x\,\partial\phi} - (A_{22}+A_{66})\sin\alpha\frac{\partial}{\partial\phi}$$

$$L^*_{21} = (A_{12}+A_{66})R(x)\frac{\partial^2}{\partial x\,\partial\phi} + (A_{22}+A_{66})\sin\alpha\frac{\partial}{\partial\phi}$$

$$L_{22}^* = A_{66}\left[R^2(x)\frac{\partial^2}{\partial x^2} + R(x)\sin\alpha\frac{\partial}{\partial x} - \sin^2\alpha\right] + A_{22}\frac{\partial^2}{\partial\phi^2}$$

$$L_{13}^* = -A_{12}R(x)\cos\alpha\frac{\partial}{\partial x} + A_{22}\sin\alpha\cos\alpha$$

$$L_{23}^* = -A_{22}\cos\alpha\frac{\partial}{\partial\phi}$$

$$L_{31}^* = -R^2(x)[A_{12}R(x)\cos\alpha\frac{\partial}{\partial x} - A_{22}\sin\alpha\cos\alpha]$$

$$L_{32}^* = -A_{22}R^2(x)\cos\alpha\frac{\partial}{\partial\phi}$$

$$L_{33}^* = D_{11}R^4(x)\frac{\partial^4}{\partial x^4} + 2(D_{12}+2D_{66})R^2(x)\frac{\partial^4}{\partial x^2\,\partial\phi^2} + D_{22}\frac{\partial^4}{\partial\phi^4}$$
$$+2D_{11}R^3(x)\sin\alpha\frac{\partial^3}{\partial x^3} - 2(D_{12}+2D_{66})R(x)\sin\alpha\frac{\partial^3}{\partial x\,\partial\phi^2}$$
$$-D_{22}R^2(x)\sin^2\alpha\frac{\partial^2}{\partial x^2} + 2(D_{12}+D_{22}+2D_{66})\sin^2\alpha\frac{\partial^2}{\partial\phi^2}$$
$$+D_{22}R(x)\sin^3\alpha\frac{\partial}{\partial x} + A_{22}R^2(x)\cos^2\alpha$$

$$L_N^* = R^3(x)\frac{\partial}{\partial x}\left[R(x)N_{x0}\frac{\partial}{\partial x}\right] + R^2(x)\frac{\partial}{\partial\phi}\left(N_{\phi 0}\frac{\partial}{\partial\phi}\right). \tag{13}$$

Now let us assume solutions for eqns (12), of the following form:

$$U = u(x)\cos n\phi, \quad V = v(x)\sin n\phi, \quad W = w(x)\cos n\phi \tag{14}$$

where

$$u(x) = \sum_{m=0}^{\infty} a_m x^m, \quad v(x) = \sum_{m=0}^{\infty} b_m x^m, \quad w(x) = \sum_{m=0}^{\infty} c_m x^m \tag{15}$$

and n is an integer representing the circumferential wave number of the buckled shell; a_m, b_m and c_m are constants to be determined later.

On substituting from eqns (14) and (15) into eqn (12) and using eqns (1) and (13) we develop three linear algebraic equations by matching the terms of same order in x, and in addition we obtain the following recurrence relations:

$$a_{m+2} = G_{1,1}a_{m+1} + G_{1,2}a_m + G_{1,3}b_{m+1} + G_{1,4}b_m + G_{1,5}c_{m+1} + G_{1,6}c_m$$
$$b_{m+2} = G_{2,1}a_{m+1} + G_{2,2}a_m + G_{2,3}b_{m+1} + G_{2,4}b_m + G_{2,5}c_m$$
$$c_{m+4} = G_{3,1}a_{m+1} + G_{3,2}a_m + G_{3,3}a_{m-1} + G_{3,4}a_{m-2} + G_{3,5}b_m + G_{3,6}b_{m-1} + G_{3,7}b_{m-2} + G_{3,8}c_{m+3}$$
$$+G_{3,9}c_{m+2} + G_{3,10}c_{m+1} + G_{3,11}c_m + G_{3,12}c_{m-1} + G_{3,13}c_{m-2} + G_{3,14}c_{m-3} \quad (m = 0,1,2,\ldots) \tag{16}$$

where the coefficients $G_{i,j}$ [$(i, j) = (1,6), (2,5)$, and $(3,14)$] are given in the Appendix. The above recurrence relations allow one to express the unknown constants a_m, b_m $(m \geqslant 2)$ and c_m $(m \geqslant 4)$ in terms of a_0, a_1, b_0, b_1, c_0, c_1, c_2 and c_3. Therefore the general form of $u(x)$, $v(x)$ and $w(x)$ may be written as

$$u(x) = u_1(x)a_0+u_2(x)a_1+u_3(x)b_0+u_4(x)b_1+u_5(x)c_0+u_6(x)c_1+u_7(x)c_2+u_8(x)c_3$$
$$v(x) = v_1(x)a_0+v_2(x)a_1+v_3(x)b_0+v_4(x)b_1+v_5(x)c_0+v_6(x)c_1+v_7(x)c_2+v_8(x)c_3$$
$$w(x) = w_1(x)a_0+w_2(x)a_1+w_3(x)b_0+w_4(x)b_1+w_5(x)c_0+w_6(x)c_1+w_7(x)c_2+w_8(x)c_3 \quad (17)$$

in which $u_i(x)$, $v_i(x)$ and $w_i(x)$ $(i = 1,2,\ldots,8)$ are the base functions of $u(x)$, $v(x)$ and $w(x)$, respectively, and a_0, a_1, b_0, b_1, c_0, c_1, c_2 and c_3 are the unknowns to be determined by imposing the boundary conditions at both ends of the cone.

Before going into details of the solution procedure, let us consider the convergence property of the series solutions $u(x)$, $v(x)$ and $w(x)$ defined in eqns (15) and the corresponding recurrence eqns (16).

Careful analysis of the recurrence eqns (16) and the coefficients $G_{i,j}$ given in the Appendix shows that:

- The power series defined in eqns (15) and (16) are alternating series, i.e. the terms of the series change sign consecutively. This property can readily be verified through numerical calculations.
- When m becomes large enough, the recurrence eqns (16) can be written approximately as follows:

$$a_{m+2} \doteq -\frac{2\sin\alpha}{R_0}a_{m+1}-\frac{\sin^2\alpha}{R_0^2}a_m$$
$$b_{m+2} \doteq -\frac{2\sin\alpha}{R_0}b_{m+1}-\frac{\sin^2\alpha}{R_0^2}b_m$$
$$c_{m+4} \doteq -\frac{4\sin\alpha}{R_0}c_{m+3}-\frac{6\sin^2\alpha}{R_0^2}c_{m+2}-\frac{4\sin^3\alpha}{R_0^3}c_{m+1}-\frac{\sin^4\alpha}{R_0^4}c_m. \quad (18)$$

These approximate recurrence equations indicate that the coefficients a_m, b_m and c_m are predominantly dependent on the former terms a_i, b_i $(i = 0, 1)$ and c_i $(i = 0, 1, 2, 3)$, respectively, when m is large enough.

Assuming the convergence ratio of $u(x)$, $v(x)$ and $w(x)$ to be ρ_a, ρ_b and ρ_c, respectively, i.e.

$$\rho_a = \lim_{m\to\infty}\frac{a_{m+1}}{a_m},\quad \rho_b = \lim_{m\to\infty}\frac{b_{m+1}}{b_m},\quad \rho_c = \lim_{m\to\infty}\frac{c_{m+1}}{c_m} \quad (19)$$

and noting the elementary character of the alternant series, eqns (18) can be changed into the following form:

$$\rho_a^2 = \frac{2\sin\alpha}{R_0}\rho_a-\frac{\sin^2\alpha}{R_0^2}$$
$$\rho_b^2 = \frac{2\sin\alpha}{R_0}\rho_b-\frac{\sin^2\alpha}{R_0^2}$$
$$\rho_c^4 = \frac{4\sin\alpha}{R_0}\rho_c^3-\frac{6\sin^2\alpha}{R_0^2}\rho_c^2+\frac{4\sin^3\alpha}{R_0^3}\rho_c-\frac{\sin^4\alpha}{R_0^4}. \quad (20)$$

It is now easy to verify, by simple manipulations, that ρ_a and ρ_b have the following identical real roots

$$\rho_a = \rho_b = \rho = \frac{\sin \alpha}{R_0} \tag{21}$$

and ρ_c has a single real root

$$\rho_c = \rho = \frac{\sin \alpha}{R_0} \tag{22}$$

therefore, the series $u(x)$, $v(x)$ and $w(x)$ obtained in the previous part have identical convergence radius r_c, i.e.

$$r_c = \frac{1}{\rho} = \frac{R_0}{\sin \alpha}. \tag{23}$$

Noting that

$$R_0 = \frac{R_1 + R_2}{2} \tag{24}$$

the convergence radius for the three series becomes

$$r_c = \frac{R_1 + R_2}{2 \sin \alpha}. \tag{25}$$

That is, as long as x is within the circle of radius r_c, convergence will be assured. For the shells considered here, the maximum value of $|x|$ is $L/2$. Thus for our purposes, the condition for convergence is

$$\frac{L}{2} \leqslant \frac{R_1 + R_2}{2 \sin \alpha}. \tag{26}$$

The convergence condition (26) can be rewritten as

$$L \sin \alpha \leqslant R_1 + R_2 \tag{27}$$

or

$$R_2 - R_1 \leqslant R_1 + R_2. \tag{28}$$

This may finally be written as

$$R_1 \geqslant 0. \tag{29}$$

Hence the three constructed series will converge to their corresponding solutions if the small radius is not zero, i.e. if the conical shell is a truncated one. A complete cone is treated as a truncated cone with a very small radius at its apex. Thus for all practical purposes, there are no limitations on the geometric parameters of the shell considered. Accordingly, the solution obtained provides exact solutions for the three displacements U, V and W for the buckling of cones under axial compressive loads and external pressure. The three displacements U, V and W may be used to calculate the stress resultants N_x, N_ϕ and $N_{x\phi}$ and the bending moments M_x, M_ϕ and $M_{x\phi}$ through eqns (6) and (7), and furthermore the transverse shear forces Q_x and Q_ϕ may be obtained from eqns (8). This solution is exact because is satisfies the governing equations rigorously and it also satisfies the eight boundary conditions through eight arbitrary constants.

The critical buckling loads and the corresponding buckling mode shapes can finally be obtained by equating the determinants of the coefficients matrix obtained after imposing the eight boundary conditions to zero.

4. NUMERICAL RESULTS AND DISCUSSIONS

4.1. *Solution procedure*

Since the solution procedure is applicable to all types of boundary conditions, a simple program, EXACT1, has been developed. The program includes the following steps:

- Input geometrical and material parameters and related coefficients;
- Calculate U, V, W, $\partial W/\partial x$, N_x, $N_{x\phi}$, Q_x and M_x for a fixed value of the critical buckling load;
- Introduce the boundary conditions and compute the determinant for specific boundary condition;
- Check for convergence.

In the programme, the buckling loads for axisymmetrical cases may be calculated directly by setting the circumferential wavenumber to zero. The buckling loads for asymmetrical cases may be obtained by minimizing the loads with respect to the circumferential wavenumber.

4.2. *Numerical results for isotropic cones*

In this section numerical results are presented for the buckling of isotropic conical shells under axial compression with different parameters and under different boundary conditions. Before presenting the results, let us introduce the following notation:

$$\rho_{cr} = \frac{P_{cr}}{P_{cl}} \tag{30}$$

where P_{cr} is the critical buckling load obtained from the present method, and P_{cl} is the classical value of the critical buckling load

$$P_{cl} = \frac{2\pi E h^2 \cos^2 \alpha}{\sqrt{3(1-\mu^2)}} \tag{31}$$

suggested by Seide (1956).

The present values P_{cr} and their comparison with those in Baruch *et al.* (1970) are shown for isotropic cones with different values of L/R_1, semi-vertex angles α and different boundary conditions, i.e. SS_1 and SS_3 in Table 1, SS_2 in Table 2, SS_4 in Table 3 and CC_1 and CC_3 in Table 4. Good agreement for ρ_{cr} can be observed between the present results and those from Baruch *et al.* (1970). There is however a difference in the circumferential wavenumber. It can be seen that ρ_{cr} tends to 0.5 for SS_1, SS_2 and SS_3 and to 1.0 for SS_4, CC_1 and CC_3. This means that there exists a lower critical value for SS_1, SS_2 and SS_3 and

Table 1. Critical load ratio ρ_{cr} for SS_3 and SS_1 boundary conditions ($R_1/h = 100.0$ $\mu = 0.3$)

L/R_1	0.2	0.2	0.5	0.5
α	present	Baruch (1970)	present	Baruch (1970)
1°	0.5032	0.4991	0.5131	0.5131
5°	0.5057	0.5021	0.5142	0.5139
10°	0.5106	0.5075	0.5151	0.5147
20°	0.5280	/	0.5163	/
30°	0.5616	0.5567	0.5140	0.5139
45°	0.6491	/	0.4947	/
60°	0.8715	0.8701	0.4486	0.4486
70°	1.2346	/	0.4304	/
80°	2.3832	2.3830	0.5405	0.5407

Table 2. Critical load ratio ρ_{cr} for SS_2 boundary condition ($R_1/h = 100.0 \quad \mu = 0.3$)

L/R_1	0.2	0.2	0.5	0.5
α	present	Baruch (1970)	present	Baruch (1970)
1°	0.5081 (1)	0.5106 (2)	0.5147 (1)	0.5191 (2)
5	0.5098 (1)	0.5133 (2)	0.5153 (1)	0.5196 (2)
10°	0.5102 (1)	0.5184 (2)	0.5163 (1)	0.5203 (2)
20°	0.5284 (1)	/	0.5179 (1)	/
30°	0.5604 (1)	0.5696 (2)	0.5166 (1)	0.5203 (2)
45°	0.6534 (1)	/	0.4992 (1)	/
60°	0.8759 (1)	0.8924 (2)	0.4596 (1)	0.4652 (2)
70°	1.2428 (1)	/	0.4423 (1)	/
80°	2.3997 (1)	2.4470 (2)	0.5572 (1)	0.5984 (2)

Table 3. Critical load ratio ρ_{cr} for SS_4 boundary condition ($R_1/h = 100.0 \quad \mu = 0.3$)

L/R_1	0.2	0.2	0.5	0.5
α	present	Baruch (1970)	present	Baruch (1970)
1°	1.0051 (7)	1.005 (7)	1.0020 (8)	1.002 (8)
5°	1.0057 (7)	1.006 (7)	1.0018 (8)	1.002 (8)
10°	1.0071 (7)	1.007 (7)	1.0012 (8)	1.002 (8)
20°	1.0097 (6)	/	1.0000 (8)	/
30°	1.0171 (5)	1.017 (5)	1.0006 (7)	1.001 (7)
45°	1.0415 (2)	/	1.0110 (5)	/
60°	1.1443 (0)	1.144 (0)	1.0032 (5)	1.044 (7)
70°	1.4207 (0)	/	1.0150 (5)	/
80°	2.4774 (0)	2.477 (0)	1.0111 (3)	1.015 (5)

Table 4. Critical load ratio ρ_{cr} by present method

L/R_1	0.2	0.5	0.2	0.5
α	CC_1	CC_1	CC_3	CC_3
1°	1.664 (0)	1.004 (8)	/	1.053 (8)
5°	1.678 (0)	1.007 (8)	1.687 (0)	1.054 (8)
10°	1.708 (0)	1.006 (8)	1.709 (0)	1.064 (8)
30°	1.945 (0)	1.002 (7)	1.947 (0)	1.099 (8)
45°	2.372 (0)	0.999 (5)	2.375 (0)	1.029 (0)
60°	3.320 (0)	1.001 (2)	3.328 (0)	1.015 (0)

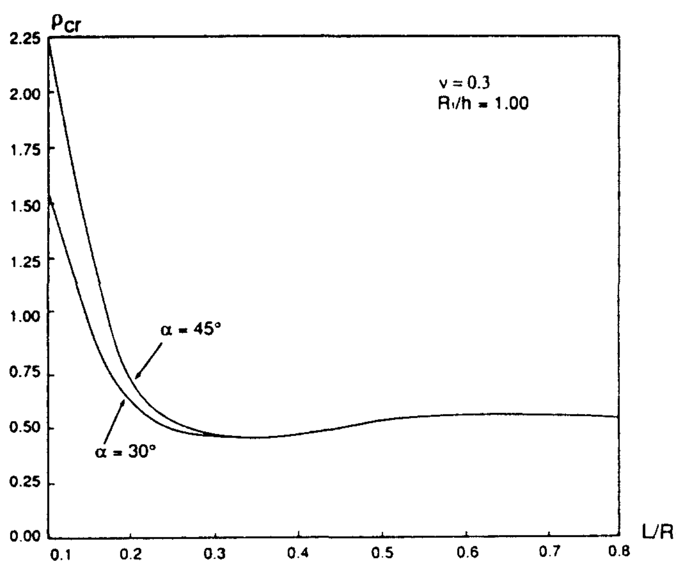

Fig. 2. Influence of L/R_1 on ratio ρ_{cr} for SS_3.

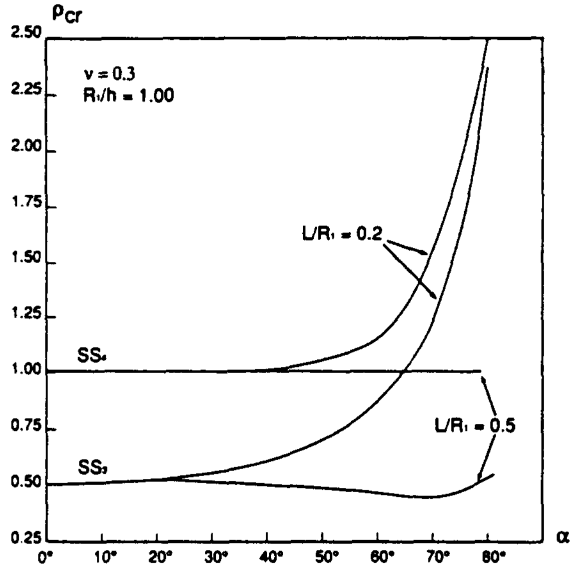

Fig. 3. Influence of α on ratio ρ_{cr} for SS_3.

Seide's formula is only applicable to SS_4, CC_1 and CC_3. For extremely short cones with $L/R_1 = 0.2$, ρ_{cr} becomes larger as α increases; and ρ_{cr} tends to a constant independent of α for cones with L/R_1 larger than 0.5. These properties are shown in Fig. 2 for SS_3 and Fig. 3 for SS_3 and SS_4. Another important phenomenon worth noting is that the buckling wavenumber tends to decrease as α increases.

All calculations show that only 20 terms of the series (14) are sufficient for accurate value of ρ_{cr}. Further calculations using 15 terms show little difference in the results obtained.

4.3. *Numerical results for orthotropic cones*

For orthotropic cones, we compute ρ_{cr} from eqn (31) with μ replaced by $\mu_{x\phi}$ and E replaced by E_x.

Numerical results for orthotropic cones with SS_3 are shown in Fig. 4, from which the influence of L/R_1 on ρ_{cr} for cones with $E_x/E_\phi = 10.0$ may be noted. it can be seen that ρ_{cr} is independent of L/R_1 when L/R_1 is larger than 1.0, and also ρ_{cr} first decreases and then increases as L/R_1 increases from 0.2 to 1.0. This curve possesses the same variation as that shown in Fig. 2.

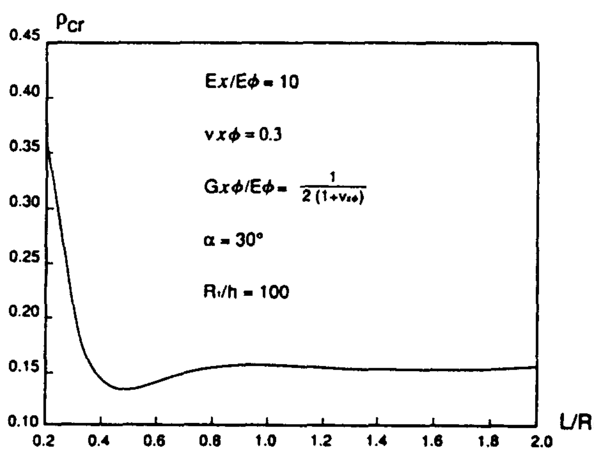

Fig. 4. Influence of L/R_1 on ratio ρ_{cr} for SS_3.

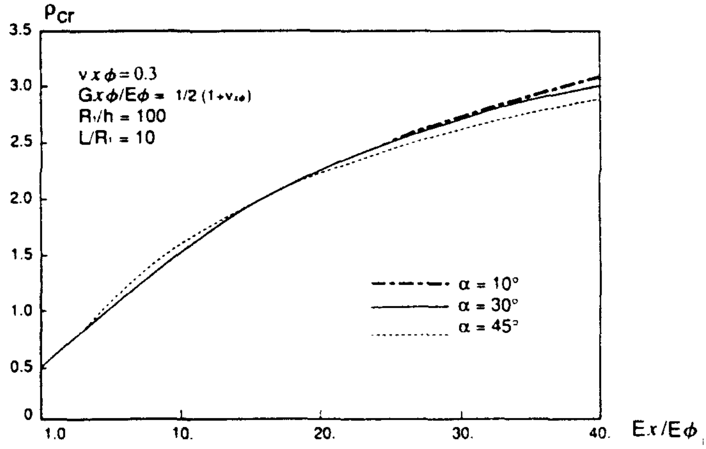

Fig. 5. Influence of Ex/E_ϕ on ratio ρ_{cr} for SS_1.

Figure 5 shows the effect of E_x/E_ϕ on ρ_{cr} for orthotropic cones with parameters given in the figure. It can be observed that ρ_{cr} increases as E_x/E_ϕ becomes large, and it approaches a constant when E_x/E_ϕ is large enough. Also of interest to note is that the semi-vertex angle α has a slight effect on ρ_{cr}, and there exists only a slight difference among ρ_{cr} for $\alpha = 10°$, 30° and 45°.

5. CONCLUSIONS

The salient points in this study include: (1) Derivation of a systematic solution procedure for buckling analysis of isotropic and orthotropic conical shells under axial compression and external pressure, using the power series method; (2) The solutions are applicable to all types of boundary conditions and to various kinds of truncated conical shells; (3) The effects of semi-vertex angle and material constants on the buckling loads are identified.

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APPENDIX

$$G_{1,1} = -\frac{(2m+1)\sin\alpha}{(m+2)R_0}$$

$$G_{1,2} = -\frac{m^2\sin^2\alpha}{(m+2)(m+1)R_0^2} + \frac{A_{22}\sin^2\alpha + A_{66}n^2}{A_{11}R_0^2(m+2)(m+1)}$$

$$G_{1,3} = -\frac{(A_{12}+A_{66})n}{A_{11}R_0(m+2)}$$

$$G_{1,4} = -\frac{[(A_{12}+A_{66})m-(A_{22}+A_{66})]n\sin\alpha}{A_{11}R_0^2(m+2)(m+1)}$$

$$G_{1,5} = \frac{A_{12}\cos\alpha}{A_{11}R_0(m+2)}$$

$$G_{1,6} = \frac{(A_{12}m-A_{22})\sin\alpha\cos\alpha}{A_{11}R_0^2(m+2)(m+1)}$$

$$G_{2,1} = \frac{(A_{12}+A_{66})n}{A_{66}R_0(m+2)}$$

$$G_{2,2} = \frac{[(A_{12}+A_{66})m+(A_{22}+A_{66})]n\sin\alpha}{A_{66}R_0^2(m+2)(m+1)}$$

$$G_{2,3} = G_{1,1}$$

$$G_{2,4} = -\frac{m^2\sin^2\alpha}{(m+2)(m+1)R_0^2} + \frac{A_{66}\sin^2\alpha + A_{22}n^2}{A_{66}R_0^2(m+2)(m+1)}$$

$$G_{2,5} = -\frac{A_{22}n\cos\alpha}{A_{66}R_0^2(m+2)(m+1)}$$

$$G_{3,1} = \frac{A_{12}\cos\alpha}{D_{11}R_0(m+4)(m+3)(m+2)}$$

$$G_{3,2} = \frac{(3A_{12}m+A_{22})\sin\alpha\cos\alpha}{D_{11}R_0^2m_{41}}$$

$$G_{3,3} = \frac{[3A_{12}(m-1)+2A_{22}]\sin^2\alpha\cos\alpha}{D_{11}R_0^3m_{41}}$$

$$G_{3,4} = \frac{[A_{12}(m-2)+A_{22}]\sin^3\alpha\cos\alpha}{D_{11}R_0^4m_{41}}$$

$$G_{3,5} = \frac{A_{22}n\cos\alpha}{D_{11}R_0^2m_{41}}$$

$$G_{3,6} = \frac{A_{22}n\sin 2\alpha}{D_{11}R_0^3m_{41}}$$

$$G_{3,7} = \frac{A_{22}n\sin^2\alpha\cos\alpha}{D_{11}R_0^4m_{41}}$$

$$G_{3,8} = -\frac{2(2m+1)\sin\alpha}{R_0(m+4)}$$

$$G_{3,9} = \left[\frac{2(D_{12}+2D_{66})n^2+D_{22}\sin^2\alpha}{D_{11}R_0^2} - \frac{P}{2\pi D_{11}R_0\cos\alpha} + \frac{6m^2\sin^2\alpha}{R_0^2}\right]\Big/(m+4)(m+3)$$

$$G_{3,10} = \left\{-\frac{q(m+1)\tan\alpha}{D_{11}} - \frac{3Pm\sin\alpha}{2D_{11}R_0^2\pi\cos\alpha} - \frac{2m(m-1)(2m-1)\sin^3\alpha}{R_0^3}\right.$$
$$\left. + \frac{[2(D_{12}+2D_{66})n^2+D_{22}\sin^2\alpha](2m-1)\sin\alpha}{D_{11}R_0^3}\right\}\Big/(m+4)(m+3)(m+2)$$

$$G_{3,11} = \Big\{[2(D_{12}+2D_{66})n^2+D_{22}\sin^2\alpha]\sin^2\alpha m(m-2) - A_{22}R_0^2\cos^2\alpha - D_{11}m(m-1)^2(m-2)\sin^4\alpha - D_{22}n^4$$
$$+\frac{qn^2R_0^3}{\cos\alpha} + 2(D_{12}+D_{22}+2D_{66})n^2\sin^2\alpha - 4qR_0^3m\tan\alpha\sin\alpha$$
$$\left. -\frac{3PR_0m(m-1)\sin^2\alpha}{2\pi\cos\alpha} - 3.5qR_0^3m(m-1)\tan\alpha\sin^2\alpha\right\}\Big/(D_{11}R_0^4m_{41})$$

$$G_{3,12} = \left[-\left(\frac{P}{2\pi \cos \alpha} + \frac{4.5qR_0^2}{\cos \alpha} \right)(m-1)(m-2) \sin^3 \alpha - 6qR_0^2(m-1) \tan \alpha \sin^2 \alpha \right.$$

$$\left. -2A_{22}R_0 \sin \alpha \cos^2 \alpha + \frac{3qR_0^2 n^2 \sin \alpha}{\cos \alpha} \right] / (D_{11}R_0^4 m_{41})$$

$$G_{3,13} = [-0.5q \tan \alpha R_0(m-2)(5m-7) \sin^3 \alpha + 3qn^2 R_0 \tan \alpha \sin \alpha - A_{22} \sin^2 \alpha \cos^2 \alpha] / (D_{11}R_0^4 m_{41})$$

$$G_{3,14} = \left[-0.5q(m-2)(m-3) \tan \alpha \sin \alpha - \frac{qn^2}{\cos \alpha} \right] \sin^3 \alpha / (D_{11}R_0^4 m_{41})$$

where $m_{41} = (m+4)(m+3)(m+2)(m+1)$.